

AP Calculus BC**Section 4.4 – Optimization**

1. An open top box is to be made by cutting small congruent squares form the corners of a 10-by-10-in. sheet of tin and bending up the sides. How large should the squares cut from the corners be to maximize the volume of the box? What is the volume?
2. A rectangle is to be inscribed in a semicircle of radius 2. What are the dimensions of the rectangle with largest area?
3. A manufacturer wants to design an open box having a square base and a surface area of 108 sq. in. What dimensions will produce a box with maximum volume?
4. A rancher has 200 feet of fencing with which to enclose two adjacent rectangular corrals. What dimensions should be used so that the enclosed area will be a maximum?
5. An open box is to be made from a rectangular piece of material by cutting equal squares from each corner and turning up the sides. Find the height of the box of maximum volume if the material has dimensions of 2 ft. by 3 ft.
6. An open box with a square base is to be constructed from 42 square meters of material. What are the dimensions if the volume is to be a maximum?
7. Find the radius of the largest right circular cylinder that can be inscribed in a sphere with $R = 100$.
8. Find the point on the graph of $y = x^2$ that is closest to the point $(3,0)$.
9. A rectangular area of 4800 square ft. is to be fenced off using two types of fencing that will be used on opposite sides of the rectangle. One type costs \$4 per foot and the other costs \$6 per foot. How much of each type should be used in order to minimize the cost of the fencing?
10. A cylindrical can (closed at the top) holds 750 ml of liquid. Find the height and radius of the can that will minimize the amount of material used in making the can.
11. A closed rectangular box with a square base is to have a volume of 2250 cubic inches. The material for the top and bottom of the box will cost \$2 per square inch, and the material for the sides will cost \$3 per square inch. Find the dimensions of the container of least cost.
12. What points on the ellipse $x^2 + 4y^2 = 8$ are nearest the point $(1, 0)$?
13. Find the height and radius of the largest cylinder that can be inscribed in a sphere of radius $R = 30$.
14. A rectangular sheet of metal is to be made into a trough by bending it so that the cross section has a U-shape that is flat on the bottom. If the metal is 10 inches wide, how deep must the trough be to carry the most water?

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15. An offshore oil well is located in the ocean at a point W , which is 6 mi. from the closest shore point A on a straight shoreline. The oil is to be piped to a shore point B that is 10 mi. from A by piping it on a straight line under water from W to some point P between A and B and then on to B via a pipe along the shoreline. If the cost of laying pipe is \$100,000 per mile under water and \$75,000 per mile over land, where should the point P be located to minimize the cost?
16. A man is on the bank of a river that is 1 mi. wide. He wants to travel to a town on the opposite bank, but it is also 1 mi. upstream. He intends to row on a straight line to some point P on the opposite bank and then walk the remaining distance along the bank. To what point should he row in order to reach his destination in the least time if he can walk 5 mph and row 3 mph?
17. Using problem #15 as the set up, with the distance from the well to the shore as 4 mi. and the distance from A to B as 8 mi., find the point on the shore that will minimize the cost.
18. Using the same set up as #16, to what point should he row in order to reach his destination in the least time if he can walk 5 mph and row 4 mph?
19. A firm determines that x units of its product can be sold daily at p dollars per unit, where $x = 1000 - p$. The cost of producing x units per day is $C(x) = 3000 + 20x$. Find the revenue and profit functions, and assuming that the production capacity is at most 500 units per day, determine how many units the company must produce and sell to maximize profit. What is the maximum profit?
20. In a certain chemical manufacturing process, the daily weight y of defective chemical output depends on the total weight x of all output according to the formula $y = 0.01x + 0.00003x^2$ where x and y are in pounds. If the revenue is \$100 per pound of nondefective chemical produced and the loss is \$20 per pound of defective chemical produced, how many pounds of chemical should be produced daily to maximize the profit?
21. A company estimates that it can sell 1000 units per week if it sets the unit price at \$3, but that its weekly sales will rise by 100 units for each \$0.10 decrease in price. Find the number of units that will maximize revenue.
22. A merchant finds that he can sell 4000 yards of a particular fabric each month if he prices it at \$6 per yard, and that his monthly sales will increase by 250 yards for each \$0.15 reduction in the price per yard. Find the price per yard that will maximize revenue.
23. The manufacturer of Zbars estimates that 100 units per month can be sold if the unit price is \$250 and that sales will increase by 10 units for each \$5 decrease in price. The fixed monthly cost of operating a plant is \$7000 while the cost of manufacturing each unit is \$100. Find the number of units sold that will maximize profit.

SECTION 4.4 - OPTIMIZATION

① a) $x+y=20$

$$S = x^2 + y^2 \\ = x^2 + (20-x)^2$$

$$S' = 2x + 2(20-x)(-1) \\ = 2x + 2x - 40 \\ = 4x - 40 = 0 \quad \Rightarrow x=10 \text{ is min}$$

$$x=10$$

CHECK ENDPOINTS: #'s ARE 0 AND 20.

b) $P = xy = x(20-x) = 20x - x^2$

$$P' = 20 - 2x = 0$$

$$x=10$$

$$P'' = -2 < 0 \Rightarrow \text{MAX AT } x=10$$

THE #'S ARE 10 AND 10.

② $P = 2x+2y \quad xy=16 \Rightarrow y = \frac{16}{x}$

$$P = 2x + 32x^{-1}$$

$$P' = 2 - 32x^{-2} = 0$$

$$2 = \frac{32}{x^2} \Rightarrow x=4$$

$$P'' = 64x^{-3}$$

$$P''(4) = 1 > 0 \Rightarrow x=4 \text{ is min}$$

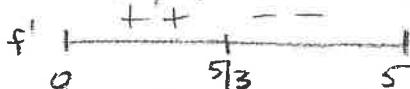
$$P = 2(4) + \frac{32}{4} = 16 \text{ min.}$$

③ $V = x(10-2x)(10-2x)$
 $= 4x(5-x)^2$

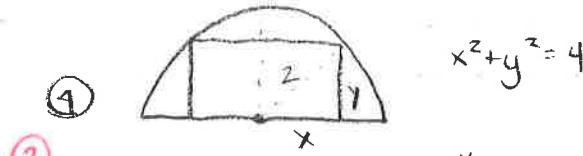
④ $V' = 4x \cdot 2(5-x)(1) + (5-x)^2 \cdot 4$
 $= 4(5-x)[-2x+5-x]$
 $= 4(5-x)(5-3x) = 0$

$$x = 5, \frac{5}{3}$$

$$D = (0, 5)$$



SQUARE SHOULD BE $\frac{5}{3}'' \times \frac{5}{3}''$



$$x^2 + y^2 = 4$$

$$A = 2xy = 2x(4-x^2)^{1/2}$$

$$A' = 2x \cdot \frac{1}{2}(4-x^2)^{-1/2}(-2x) + (4-x^2)^{1/2}$$

$$= 2(4-x^2)^{-1/2}[-x^2 + 4 - x^2]$$

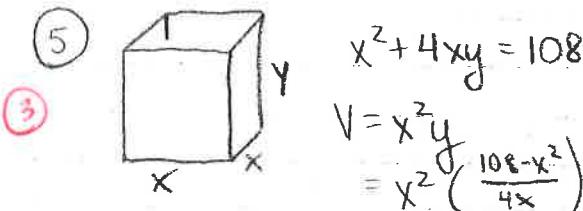
$$= 2(4-x^2)^{-1/2}(4-2x^2)$$

$$= 4(4-x^2)^{-1/2}(2-x^2) = 0$$

$$x = \sqrt{2} \quad \begin{matrix} + & + & - & - \end{matrix}$$

$$D: (0, 2) \quad \begin{matrix} \leftarrow & \frac{1}{\sqrt{2}} & \rightarrow \end{matrix} \quad 2$$

DIMENSIONS ARE $2\sqrt{2} \times \sqrt{2}$



$$x^2 + 4xy = 108$$

$$V = x^2 y$$

$$= x^2 \left(\frac{108-x^2}{4x} \right)$$

$$V = \frac{1}{4} (108x - x^3) \quad D: (0, \sqrt{108})$$

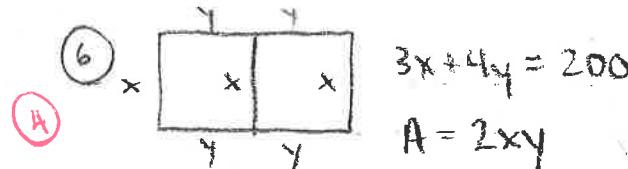
$$V' = \frac{1}{4} (108 - 3x^2) = 0 \Rightarrow x = 6$$

$$V'' = \frac{1}{4} (-6x)$$

$$V''(6) = -9 < 0 \Rightarrow \text{MAX AT } x=6.$$

$$y = \frac{108-6^2}{4 \cdot 6} = 3 \quad \text{DIMENSIONS ARE}$$

$$6'' \times 6'' \times 3''$$



$$3x + 4y = 200$$

$$A = 2xy$$

$$A = 2x \left(\frac{200-3x}{4} \right) = \frac{1}{2} (200x - 3x^2)$$

$$A' = \frac{1}{2} (200 - 6x) = 0 \Rightarrow x = 33\frac{1}{3}'$$

$$A'' = \frac{1}{2} (-6) < 0 \Rightarrow \text{MAX AT } x = 33\frac{1}{3}'$$

$$y = \frac{200-3(33\frac{1}{3}')}{4} = 25'$$

DIMENSIONS OF WHOLE RECTANGLE
 ARE $50' \times 33\frac{1}{3}'$

$$\begin{aligned}
 7) \quad V &= x(20-2x)(20-2x) \\
 &= 4x(10-x)^2 \\
 V' &= 4x \cdot 2(10-x)(-1) + (10-x)^2(4) \\
 &= 4(10-x)[-2x+10-x] \\
 &= 4(10-x)(10-3x) = 0 \Rightarrow x = \frac{10}{3} \\
 D: (0, 10) &\quad \begin{array}{c} ++ \\ \hline 0 \quad \frac{10}{3} \quad 10 \\ - - \end{array} \\
 V &= 4\left(\frac{10}{3}\right)\left(10-\frac{10}{3}\right)^2 = 592.59 \text{ m}^3
 \end{aligned}$$

$$\begin{aligned}
 11) \quad A &= xy \quad 2x+2y = 500 \\
 &= x(250-x) \quad x+y = 250 \\
 &= 250x-x^2 \\
 A' &= 250-2x = 0 \Rightarrow x = 125 \\
 A'' &= -2 < 0 \Rightarrow \text{MAX} \rightarrow \\
 \text{DIMENSIONS ARE } &125 \text{ m} \times 125 \text{ m}
 \end{aligned}$$

$$\begin{aligned}
 12) \quad P &= 2x+2y \quad xy = 100 \\
 &= 2x+2\left(\frac{100}{x}\right) \\
 &= 2x+\frac{200}{x} \quad \text{DIMENSIONS ARE} \\
 P' &= 2-200x^{-2}=0 \quad 10 \text{ m} \times 10 \text{ m} \\
 x &= 10 \\
 P'' &= 400x^{-3} \\
 P''(10) &> 0 \Rightarrow \text{MIN}
 \end{aligned}$$

$$\begin{aligned}
 8) \quad V &= x(2-2x)(3-2x) \quad D: (0, 1) \\
 &= 2x(1-x)(3-2x) \\
 &= 2x(2x^2-5x+3) \\
 &= 2(2x^3-5x^2+3x) \\
 V' &= 2(6x^2-10x+3)=0 \\
 x &= \frac{10 \pm 2\sqrt{7}}{12} = \frac{5 \pm \sqrt{7}}{6} \quad \text{using } \frac{5-\sqrt{7}}{6} \\
 V'' &= 2(12x-10) \approx .392 \\
 V''(.392) &= -10.58 < 0 \Rightarrow \text{MAX} \rightarrow
 \end{aligned}$$

$$\begin{aligned}
 9) \quad &\text{Diagram of a rectangle with width } x \text{ and height } y \\
 &2x+y = 190 \\
 &A = xy \\
 &A = 190x-2x^2 \\
 A' &= 190-4x = 0 \Rightarrow x = 47.5 \\
 A'' &= -4 < 0 \Rightarrow \text{MAX} \\
 \text{DIMENSIONS: } &47.5' \times 95'
 \end{aligned}$$

$$\begin{aligned}
 10) \quad &x+2y=120 \\
 P &= (120-2y)y = 120y-2y^2 \\
 P' &= 120-4y=0 \\
 y &= 30 \\
 P'' &= -4 < 0 \Rightarrow \text{MAX AT } y=30 \\
 D: (0, 60) & \\
 \text{THE #'S ARE } &x=60 \text{ AND } 30
 \end{aligned}$$

$$\begin{aligned}
 13) \quad &\text{Diagram of a rectangular prism with dimensions } x, y, z \\
 &x^2+4xy=42 \\
 &y = \frac{42-x^2}{4x} \\
 V &= x^2y = x^2\left(\frac{42-x^2}{4x}\right) = \frac{1}{4}(42x-x^3) \\
 V' &= \frac{1}{4}(42-3x^2)=0 \\
 x &= \sqrt{14} \approx 3.74 \\
 V'' &= 6x^2 > 0 \Rightarrow \text{MAX} \\
 \text{DIMENSIONS: } &3.74 \text{ m} \times 3.74 \text{ m} \times 1.87 \text{ m} \left(\frac{\sqrt{14}}{2}\right)
 \end{aligned}$$

$$\begin{aligned}
 14) \quad &\text{Diagram of a cylinder with radius } r \text{ and height } h \\
 &r^2+h^2=100^2 \\
 &r^2=100^2-h^2 \\
 V &= \pi r^2(2h) \\
 &= \pi(100^2-h^2)(2h) \\
 &= 2\pi(100^2h-h^3) \\
 V' &= 2\pi(100^2-3h^2)=0 \\
 h^2 &= \frac{100^2}{3} \Rightarrow h = \frac{100}{\sqrt{3}} \quad 2h = \frac{200}{\sqrt{3}} \approx 115.470 \\
 V'' &= 2\pi(-6h) < 0 \Rightarrow \text{MAX} \\
 r^2 &= 100^2 - \frac{100^2}{3} \\
 r &= 81.650 \quad \left(\frac{100\sqrt{2}}{\sqrt{3}}\right)
 \end{aligned}$$

Section 4.4 Worksheet (cont)

$$(15) D^2 = (x-3)^2 + y^2 \\ = (x-3)^2 + \frac{y^2}{4}$$

$$(6) (D^2)' = 2(x-3) + 4x^3 = 0 \\ = 4x^3 + 2x - 6 = 0$$

$$x=1$$

$$(D^2)'' = 12x + 2 > 0 \text{ at } x=1 \Rightarrow \text{min.} \\ \text{point is } (1,1)$$

$$(16) C = 8x + 12y \quad 4800 = xy \\ C = 8x + 12\left(\frac{4800}{x}\right)$$

$$C' = 8 - 12\left(\frac{4800}{x^2}\right) = 0 \\ 8x^2 = 12(4800)$$

$$x = 60\sqrt{2} \approx 84.853 \text{ ft or } 44$$

$$y = \frac{4800}{60\sqrt{2}} = 40\sqrt{2} \approx 56.569 \text{ ft or } \$1.6$$

$$(17) \pi r^2 h = 750$$

$$(18) S = 2\pi r^2 + 2\pi rh \\ = 2\pi r^2 + 2\pi r\left(\frac{750}{\pi r^2}\right)$$

$$S' = 4\pi r - 1500; r^2 > 0$$

$$4\pi r^2 - 1500 = 0$$

$$r \approx 4.924 \text{ cm}$$

$$h = \frac{750}{\pi r^2} = 9.847 \text{ cm}$$

$$(19) x^2 y = 2250 \Rightarrow y = \frac{2250}{x^2}$$

$$(11) C = 4x^2 + 12xy \\ C = 4x^2 + 12x\left(\frac{2250}{x^2}\right)$$

$$C' = 8x - 12(2250)x^{-2} = 0$$

$$8x^3 - 12(2250) = 0$$

$$x = 15 \text{ in}$$

$$y = 10 \text{ in}$$

$$(19) D^2 = (x-1)^2 + y^2 \\ = (x-1)^2 + \left(\frac{8-x^2}{4}\right) \\ (D^2)' = 2(x-1) + \frac{1}{4}(-2x) = 0 \\ 2x-2 - \frac{1}{2}x = 0 \\ \frac{3}{2}x = 2 \\ x = \frac{4}{3}$$

$$(D^2)'' = 2 - \frac{1}{2} > 0 \text{ min.}$$

$$\left(\frac{4}{3}, \pm \frac{\sqrt{14}}{3}\right)$$

$$x = 4\sqrt{3}$$

$$(D^2)'' = 2 - \frac{1}{2} > 0 \text{ min.}$$

$$(20) r^2 + h^2 = 900$$

$$(13) V = 2\pi r^2 h \\ = 2\pi(900 - h^2)h \\ = 2\pi(900h - h^3)$$

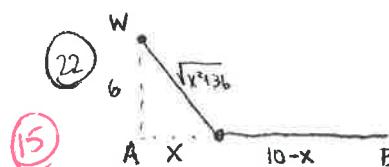
$$V' = 2\pi(900 - 3h^2) = 0$$

$$h = 10\sqrt{3}$$

$$2h = 20\sqrt{3} \approx 34.641$$

$$r^2 = 900 - 300 \Rightarrow r = 10\sqrt{6} \approx 24.495$$

$$(21) 2x + y = 10 \\ A = x(10 - 2x) = 10x - 2x^2 \\ A' = 10 - 4x = 0 \\ x = \frac{10}{4} = 5/2 = 2.5 \text{ in.} \\ y = 5 \text{ in.}$$



$$(22) C = 100\sqrt{x^2 + 36} + 75(10-x) \\ C' = 50(x^2 + 36)^{-1}(2x) + 75(-1) = 0$$

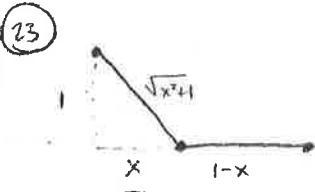
$$\frac{100x}{\sqrt{x^2 + 36}} = 75$$

$$\frac{4}{3}x = \sqrt{x^2 + 36}$$

$$\frac{16}{9}x^2 = x^2 + 36$$

$$\frac{7}{9}x^2 = 36$$

$$x = 18/\sqrt{7} \approx 6.803 \text{ mi}$$



$$T = \frac{\sqrt{x^2+1}}{3} + \frac{(1-x)}{5}$$

$$T' = \frac{1}{3} \cdot \frac{1}{(x^2+1)^{3/2}} (2x) + 4(-1)$$

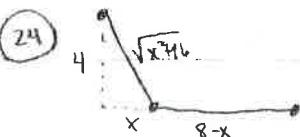
$$\frac{x}{3\sqrt{x^2+1}} = \frac{1}{5}$$

$$3\sqrt{x^2+1} = 5x$$

$$9x^2 + 9 = 25x^2$$

$$9 = 16x^2$$

$$\sqrt[3]{\frac{9}{16}} = x$$



$$C = 100\sqrt{x^2+64} + 75(8-x)$$

$$C' = 50(x^2+64)^{-1/2}(2x) - 75 = 0$$

$$\frac{100x}{\sqrt{x^2+64}} = 75$$

$$100x/75 = \sqrt{x^2+64}$$

$$\frac{16x^2}{9} = x^2 + 64$$

$$\frac{8x^2}{9} = 16 \Rightarrow x = 4.536$$

$$x = \sqrt{16} = 4\text{ mi}$$

(25) $T = \frac{\sqrt{x^2+1}}{4} + \frac{(1-x)}{5}$

$$T' = \frac{1}{4} \cdot \frac{1}{(x^2+1)^{3/2}} (2x) - \frac{1}{5} = 0$$

$$\frac{x}{4\sqrt{x^2+1}} = \frac{1}{5}$$

$$4\sqrt{x^2+1} = 5x$$

$$16(x^2) + 16 = 25x^2$$

$$16 = 9x^2$$

$\sqrt{16} = x \Rightarrow$ ROW DIRECTLY TO TOWN.

(26) $r(x) = x(1000-x) = 1000x - x^2$

(19) $P(x) = 1000x - x^2 - 3000 - 20x$
 $= -x^2 + 980x - 3000$

$$P'(x) = -2x + 980 = 0 \Rightarrow x = 490$$

$$P''(x) = -2 \Rightarrow \text{MAX}$$

PROFIT $\approx \$237,100$

(27) $P = 100(x - .01x - .00003x^2) - 20(.01x + .00003x^2)$

$$= -.0036x^2 + 98.8x$$

$$P' = -.0072x + 98.8 = 0 \Rightarrow x = 13722,$$

$$P'' = -.0072 < 0 \Rightarrow \text{MAX}$$

(28)

# UNITS	PRICE
1000	\$3
1100	2.90

 $y - 3 = \frac{-1}{100}(x - 1000)$

(29) $y = .001x + 4$

$$r(x) = x(-.001x + 4) = -.001x^2 + 4x$$

$$r'(x) = -.002x + 4 = 0$$

$$x = 2000$$

$$r''(x) = -.002 < 0 \Rightarrow \text{MAX}$$

(29)

# UNITS	PRICE
4000	6
4250	5.85

 $y - 6 = \frac{-15}{250}(x - 4000)$

(22) $y = -.0006x + 8.4$

$$r(x) = x(-.0006x + 8.4)$$

$$= -.0006x^2 + 8.4x$$

$$r'(x) = -.0012x + 8.4$$

$$x = 7000$$

$$y = -.0006(7000) + 8.4$$

$$= \$4.20$$

(30)

# UNITS	PRICE
100	250
110	245

 $y - 250 = \frac{-5}{10}(x - 100)$

(23) $y = \frac{1}{2}x + 300$

$$r(x) = x(300 - \frac{1}{2}x) = 300x - \frac{1}{2}x^2$$

$$c(x) = 7000 + 100x$$

$$p(x) = 300x - \frac{1}{2}x^2 - 7000 - 100x$$

$$= -\frac{1}{2}x^2 + 200x - 7000$$

$$p'(x) = -x + 200 = 0 \Rightarrow x = 200$$